An explicit computation of the Bures metric over the space of *N*-dimensional density matrices

An explicit computation of the Bures metric over the space of N-dimensional density matrices

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Abstract
The aim of this paper is to provide a method for explicit computation of the
Bures metric over the space of N-level quantum system, based on the coset
parametrization of density matrices.

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1. Introduction

Recent developments in the emerging field of quantum information theory have received a
great deal of attention in investigation of the properties of the set of density matrices of an
N-level quantum system. In view of such considerable interest a lot of work has been devoted
to describe and parametrize density matrices. Any density matrix of an N-level quantum
system can be expanded in terms of $N^2 - 1$ orthogonal generators of $SU(N)$ [1], which is a
generalization of the Bloch or coherence vector representation for two-level systems. Boya
et al [2] have shown that the mixed state density matrices for N-level quantum systems can
be parametrized in terms of squared components of an $(N - 1)$-sphere and unitary matrices.
By using the Euler angle parametrization of an $SU(3)$ group [3], Byrd and Slater [4] have
presented a parametrization for density matrices of three-level systems. An Euler angle-based
parametrization for density matrices of four-level systems is also introduced in [5], and has
been used by Tilma and Sudarshan [6] in order to study the entanglement properties of the two-
qubit system. A generalized Euler angle parametrization for $SU(N)$ and $U(N)$ groups is given
by Tilma and Sudarshan [7, 8]. In a comprehensive analysis [9], Życzkowski and Słomczyński
analyzed the geometrical properties of the set of mixed states for an arbitrary N-level system
and classified the space of density matrices. Diţă [10] has provided an explicit parametrization
for general N-dimensional Hermitian operators that may be considered either as Hamiltonian
or density matrices. The parametrization is based on the factorization of $N \times N$ unitary
matrices [11]. A parametrization useful to study the entanglement properties of two-qubit
density matrices is also introduced in [12], in which authors have shown that the space of

two-qubit density matrices can be characterized with 12-dimensional (as real manifold) space of a complex orthogonal group $SO(4, \mathbb{C})$ together with four positive Wootters's numbers [13], where, of course, the normalization condition reduces the number of parameters to 15.

Efforts have been also made to study the geometry of density matrices. In recent years, the Riemannian Bures metric [14] has become an interesting subject for the understanding of the geometry of quantum state space. It is the quantum analog of Fisher information in classical statistics, i.e. in the subspace of diagonal matrices it induces the statistical distance [15]. The Bures measure is monotone in the sense that it does not increase under the action of completely positive, trace preserving maps [16]. It is, indeed, minimal among all monotone metrics and its extension to pure state is exactly the Fubini–Study metric [16]. The Bures distance between any two mixed states ρ_1 and ρ_2 is a function of their fidelity $F(\rho_1, \rho_2)$ [17, 18]

$$d_B(\rho_1, \rho_2) = \sqrt{2 - 2\sqrt{F(\rho_1, \rho_2)}}, \qquad F(\rho_1, \rho_2) = \left[\mathrm{Tr} \left(\sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \right) \right]^2. \tag{1}$$

Fidelity allows one to characterize the closeness of the pair of mixed states ρ_1 and ρ_2, so, it is an important concept in quantum mechanics, quantum optics and quantum information theory. An explicit formula for the infinitesimal Bures distance between ρ and $\rho + d\rho$ was found by Hübner [19]

$$d_B(\rho, \rho + d\rho)^2 = \frac{1}{2} \sum_{i,j=1}^{N} \frac{|\langle \lambda_i | d\rho | \lambda_j \rangle|^2}{\lambda_i + \lambda_j}, \tag{2}$$

where λ_j and $|\lambda_j\rangle$ ($j = 1, 2, \ldots, N$) represent eigenvalues and eigenvectors of ρ, respectively. Dittmann has derived several explicit formulae, that do not require any diagonalization procedure, for Bures metric on the manifold of finite-dimensional nonsingular density matrices [20, 21].

The probability measure induced by the Bures metric in the space of mixed quantum states has been defined by Hall [22]. The question of how many entangled or separable states are there in the set of all quantum states is considered by Życzkowski *et al* in [23, 24]. Sommers and Życzkowski [25] have computed the Bures volume of the $(N^2 - 1)$-dimensional convex set and the $(N^2 - 2)$-dimensional hyperarea of the density matrices of an N-level quantum system. In a considerable work, Slater investigated the use of the volume elements of the Bures metric as a natural measure over the $(N^2 - 1)$-dimensional convex set of N-level density matrices to determine or estimate the volume of separable states of the pairs of qubit–qubit [26, 27] and qubit–qutrit [28, 29].

By using the Dittmann formula [20] and the Euler-angle parametrization [4], Slater has computed the Bures metric for the eight-dimensional state space of the three-level quantum systems [30]. In a similar work, but instead in terms of the coset space parametrization, we have very recently given an explicit expression for the Bures metric of the space of a three-level quantum system [31]. The coset parametrization provides a geometrical description of the set of density matrices and, as well as the Euler angle parametrization does, eliminates any overparametrization of the density matrix [10].

The aim of this paper is to provide a method for computation of the Bures metric in N-level quantum systems, based on the Hübner formula and the coset parametrization of density matrices. We also use the possibility of factorizing each coset component in terms of a diagonal phase matrix and an orthogonal matrix [11, 32]. The paper, therefore, can be regarded as a further development in the explicit computation of the Bures metric. We show that in the canonical coset parametrization, the Bures metric matrix is divided into two blocks: an $(N - 1)$-dimensional diagonal matrix corresponding to the eigenvalue coordinates and an $N(N - 1)$-dimensional matrix corresponding to the coset coordinates. It therefore provides

a factorization of the Bures measure on the space of density matrices as the product of the measure on the space of eigenvalues and the truncated Haar measure on the space of unitary matrices. It is shown that the coset parametrization gives a compact expression for all metric elements. The analytical expression for Bures metric can be used in computing the Bures volumes of quantum states as well as to study the problem of what proportion of the convex set of the bipartite systems is separable. The results also enable the calculation of the minimum Bures distance of a given density matrix from the convex set of separable states, in order to quantify entanglement of the state [33, 34].

The paper is organized as follows: in section 2, the coset space parametrization of an *N*-level density matrix is introduced. Based on the parametrization, we provide in section 3 a formula for computation of the Bures metric, explicitly. The paper is concluded in section 4 with a brief conclusion.

2. Canonical coset parametrization of density matrices

The state space of an *N*-level quantum system is identified with the set of all $N \times N$ Hermitian positive semidefinite complex matrices of trace unity, and comprises an $(N^2 - 1)$-dimensional convex set $\mathcal{M}_N$. The total number of independent variables needed to parametrize a density matrix ρ is equal to $N^2 - 1$, provided no degeneracy occurs. Let us denote the set of all diagonal density matrices of an *N*-level quantum system by $\mathcal{D}_N$. An arbitrary element $\rho^{(D)} \in \mathcal{D}_N$ can be written as

$$\rho^{(D)} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}, \qquad 0 \leqslant \lambda_i \leqslant 1, \qquad \sum_{i=1}^{N} \lambda_i = 1, \qquad (3)$$

which simply denotes an $(N-1)$-dimensional simplex $\mathcal{S}_{N-1}$ for the set of all diagonal density matrices.

A generic density matrix $\rho \in \mathcal{M}_N$ in an arbitrary basis can be obtained as the orbit of points $\rho^{(D)} \in \mathcal{D}_N$ under the action of the unitary group $U(N)$ as

$$\rho = U \rho^{(D)} U^{\dagger}. \qquad (4)$$

Let *H* be a maximum stability subgroup, i.e. a subgroup of $U(N)$ that consists of all the group elements *h* that will leave the diagonal state $\rho^{(D)}$ invariant,

$$h \rho^{(D)} h^{\dagger} = \rho^{(D)}, \qquad h \in H, \qquad \rho^{(D)} \in \mathcal{D}_N, \qquad (5)$$

that is, *H* contains all elements of $U(N)$ that commute with $\rho^{(D)}$. For every element $U \in U(N)$, there is a unique decomposition of *U* into a product of two group elements, one in *H* and the other in the quotient G/H [35], i.e.

$$U = \Omega h, \qquad U \in U(N), \qquad h \in H, \qquad \Omega \in U(N)/H. \qquad (6)$$

Therefore the action of an arbitrary group element $U \in U(N)$ on the point $D \in \mathcal{D}_N$ is given by

$$\rho = U \rho^{(D)} U^{\dagger} = \Omega h \rho^{(D)} h^{\dagger} \Omega^{\dagger} = \Omega \rho^{(D)} \Omega^{\dagger}. \qquad (7)$$

Since $\mathcal{D}_N$ consists of points with different degrees of degeneracy, the maximum stability subgroup will differ for different $\rho^{(D)} \in \mathcal{D}_N$ [9]. Let m_i denotes degree of degeneracy of eigenvalue λ_i of matrix $\rho^{(D)}$. It follows from this kind of spectrum that $\rho^{(D)}$ remains invariant under the action of arbitrary unitary transformation performed in each of the m_i-dimensional eigensubspaces. Therefore $H = U(m_1) \otimes U(m_2) \otimes \cdots \otimes U(m_k)$ is a

maximum stability subgroup for $\rho^{(D)}$, and the quotient space $U(N)/H$ is a complex flag manifold

$$\mathcal{F} = \frac{U(N)}{U(m_1) \otimes U(m_2) \otimes \cdots \otimes U(m_k)}, \qquad m_1 + m_2 + \cdots + m_k = N. \quad (8)$$

Two special kinds of degeneracy of the spectrum of $\rho^{(D)}$ are as follows. (i) Let $\rho^{(D)}$ represents the maximally mixed state $\rho_* = \mathrm{diag}\{\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}\}$. In this case the stability subgroup H is $U(N)$, and the orbit of point ρ_* is only one point, i.e. $\rho = \rho_*$. (ii) On the other hand if the spectrum of $\rho^{(D)}$ is non-degenerate, then the stability subgroup is an N-dimensional torus $T^N = U(1)^{\otimes N}$, and the orbit of the point $\rho^{(D)}$ is

$$\rho = \Omega \rho^{(D)} \Omega^\dagger, \qquad \Omega \in U(N)/T^N. \quad (9)$$

Since the maximal torus T^N is itself a subgroup of all maximum stability subgroups, the orbit of points $\rho^{(D)} \in \mathcal{D}_N$ under the action of quotient $U(N)/T^N$ generates all points of the space $\mathcal{M}_N$. The diagonal matrix $\rho^{(D)}$ is defined up to a permutation of its entries and, one can divide the simplex $\mathcal{S}_{N-1}$ into $N!$ identical simplexes and take any of them. Each part identifies points of $\mathcal{S}_{N-1}$ which have the same coordinates, but with different ordering, and can be considered as the homomorphic image of simplex $\mathcal{S}_{N-1}$ relative to the discrete permutation group P_N, i.e. $\mathcal{S}_{N-1}/P_N$. Therefore the points of $\mathcal{M}_N$ can be characterized as the orbit of diagonal matrices $D \in \mathcal{S}_{N-1}/P_N$ under the action of quotient $\Omega \in U(N)/T^N$.

   Further insight into the space of density matrices can be obtained by writing the quotient $\Omega \in U(N)/T^N$ as the product of $N-1$ components as ([35], page 401)

$$\Omega = \Omega^{(N;N)} \Omega^{(N-1;N)} \cdots \Omega^{(2;N)}, \quad (10)$$

where

$$\Omega^{(m;N)} \in \frac{U(m) \otimes T^{N-m}}{U(m-1) \otimes T^{N-m+1}}, \qquad m = 2, \ldots, N. \quad (11)$$

A typical coset representative $\Omega^{(m;N)}$ can be written as

$$\Omega^{(m;N)} = \left(\begin{array}{c|c} SU(m)/U(m-1) & O \\ \hline O^T & I_{N-m} \end{array} \right), \quad (12)$$

where O, O^T and I_{N-m} represent, respectively, the $m \times (N-m)$ zero matrix, its transpose and the $(N-m) \times (N-m)$ identity matrix. The $2(m-1)$-dimensional coset space $SU(m)/U(m-1)$ has the following $m \times m$ matrix representation ([35], page 351)

$$SU(m)/U(m-1) = \left(\begin{array}{c|c} \cos\sqrt{B^{(m)}[B^{(m)}]^\dagger} & B^{(m)} \frac{\sin\sqrt{[B^{(m)}]^\dagger B^{(m)}}}{\sqrt{[B^{(m)}]^\dagger B^{(m)}}} \\ \hline -\frac{\sin\sqrt{[B^{(m)}]^\dagger B^{(m)}}}{\sqrt{[B^{(m)}]^\dagger B^{(m)}}}[B^{(m)}]^\dagger & \cos\sqrt{[B^{(m)}]^\dagger B^{(m)}} \end{array} \right), \quad (13)$$

where $B^{(m)}$ represents an $(m-1) \times 1$ complex matrix and $[B^{(m)}]^\dagger$ is its adjoint.

   Now by parametrizing the column matrix $B^{(m)}$ as $\left(\gamma_1^{(m)} e^{i\xi_1^{(m)}}, \gamma_2^{(m)} e^{i\xi_2^{(m)}}, \ldots, \gamma_{m-1}^{(m)} e^{i\xi_{m-1}^{(m)}} \right)^T$ for $m = 2, 3, \ldots, N$, where $\gamma_i^{(m)}$ and $\xi_i^{(m)}$ are real numbers[1], the component $\Omega^{(m;N)}$ can be factorized as

$$\Omega^{(m;N)} = X^{(m;N)} R^{(m;N)} X^{(m;N)\dagger} \qquad \text{for} \quad m = 2, 3, \ldots, N, \quad (14)$$

---

[1] The correspondence between general notation of this paper and that in [31], where the $N = 3$ case is explicitly computed, is as $\gamma_1^{(2)} = \alpha$, $\xi_1^{(2)} = \phi$, $\gamma_{1,2}^{(3)} = \beta_{1,2}$ and $\xi_{1,2}^{(3)} = \psi_{1,2}$.

where $X^{(m;N)}$ is a diagonal $N \times N$ phase matrix with $X_{kl}^{(m;N)} = \delta_{kl} \exp\{i\xi_k^{(m)}\}$ and $\xi_i^{(m)} = 0$ for $i \geqslant m$, and $R^{(m;N)}$ is an $N \times N$ orthogonal matrix with the following nonzero elements

$$
\begin{aligned}
R_{ij}^{(m;N)} &= \delta_{ij} + \hat{\gamma}_i^{(m)} \hat{\gamma}_j^{(m)} (\cos \gamma^{(m)} - 1) & &\text{for} \quad 1 \leqslant i, j \leqslant m - 1 \\
R_{im}^{(m;N)} &= -R_{mi}^{(m;N)} = \hat{\gamma}_i^{(m)} \sin \gamma^{(m)} & &\text{for} \quad 1 \leqslant i \leqslant m - 1 \\
R_{mm}^{(m;N)} &= \cos \gamma^{(m)} \\
R_{ii}^{(m;N)} &= 1 & &\text{for} \quad m + 1 \leqslant i \leqslant N,
\end{aligned}
\tag{15}
$$

where we have defined $\hat{\gamma}_i^{(m)} = \gamma_i^{(m)} / \gamma^{(m)}$ and $\gamma^{(m)} = \sqrt{\sum_{i=1}^{m-1} (\gamma_i^{(m)})^2}$. As we will see later the important ingredient of our approach in computing the Bures metric is the possibility of writing the factorization (14).

3. Bures metric

In this section we shall attempt to develop a method of computing the Bures metric of an arbitrary density matrix of an *N*-level quantum system. We will use the canonical coset parametrization of the density matrices introduced in the last section.

Let ρ be a generic density matrix of an *N*-level quantum system, with eigenvalues λ_j and corresponding eigenvectors $|\lambda_j\rangle$, $(j = 1, 2, \ldots, N)$. In the light of $\rho = \Omega \rho^{(D)} \Omega^\dagger$, with $\rho^{(D)} = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ as the diagonal matrix of ρ eigenvalues, the ρ eigenvectors can be written in terms of $\rho^{(D)}$ eigenvectors as $|\lambda_i\rangle = \Omega|i\rangle$. Therefore invoking the Hübner formula (2), we can write the infinitesimal Bures distance between ρ and $\rho + d\rho$ as

$$
d_B(\rho, \rho + d\rho)^2 = \frac{1}{2} \sum_{i,j=1}^{N} \frac{|(\langle i|\Omega^\dagger) \, d(\Omega \rho^{(D)} \Omega^\dagger)(\Omega|j\rangle)|^2}{\lambda_i + \lambda_j},
\tag{16}
$$

which takes the form (remember that Ω is unitary and therefore $d\Omega^\dagger = -\Omega^\dagger \, d\Omega \Omega^\dagger$)

$$
d_B(\rho, \rho + d\rho)^2 = \frac{1}{2} \sum_{i,j=1}^{N} \frac{|\langle i|d\rho^{(D)}|j\rangle + \langle i|[\Omega^\dagger \, d\Omega, \rho^{(D)}]|j\rangle|^2}{\lambda_i + \lambda_j},
\tag{17}
$$

By using the equations $\rho^{(D)}|i\rangle = \lambda_i|i\rangle$ and $\langle i|j\rangle = \delta_{ij}$ we get

$$
d_B(\rho, \rho + d\rho)^2 = \sum_{i=1}^{N} \frac{(d\lambda_i)^2}{4\lambda_i} + \sum_{i<j}^{N} \Lambda_{ij}|(\Omega^\dagger \, d\Omega)_{ij}|^2,
\tag{18}
$$

where we have defined Λ_{ij} as

$$
\Lambda_{ij} = \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j}.
\tag{19}
$$

Equation (18) shows that the infinitesimal Bures distance is divided into two infinitesimal distances corresponding to the eigenvalues coordinates and coset coordinates. Therefore the Bures metric matrix becomes block diagonal as

$$
g = \left(\begin{array}{c|c} g^{(D)} & 0 \\ \hline 0 & g^{(C)} \end{array} \right),
\tag{20}
$$

where $g^{(D)}$ is a part of the Bures metric corresponding to the diagonal density matrix $\rho^{(D)}$, and $g^{(C)}$ is the contribution of the coset coordinates in the Bures metric.

In what follows our goal is to calculate the matrix elements of $g^{(D)}$ and $g^{(C)}$. In order to calculate $g^{(D)}$, we first note that the N eigenvalues can be parametrized explicitly in terms of $N-1$ independent parameters θ_k ($k=1,2,\ldots,N-1$) as

$$\begin{cases} \lambda_k = \sin^2\theta_1 \sin^2\theta_2 \cdots \sin^2\theta_{k-1} \cos^2\theta_k & \text{for} \quad k=1,2,\ldots,N-1 \\ \lambda_N = \sin^2\theta_1 \sin^2\theta_2 \cdots \sin^2\theta_{N-1}. \end{cases} \tag{21}$$

By using the above coordinates for the eigenvalues of ρ, we can write the infinitesimal Bures distance over the eigenvalues coordinates as

$$\sum_{i=1}^{N} \frac{(\mathrm{d}\lambda_i)^2}{4\lambda_i} = \sum_{k,l=1}^{N-1} g_{kl}^{(D)} \, \mathrm{d}\theta_k \, \mathrm{d}\theta_l, \tag{22}$$

where it can be easily seen that the metric $g_{kl}^{(D)}$ is diagonal with elements

$$\begin{aligned} g_{\theta_1\theta_1}^{(D)} &= 1 \\ g_{\theta_k\theta_k}^{(D)} &= \sin^2\theta_1 \sin^2\theta_2 \cdots \sin^2\theta_{k-1}, \qquad \text{for} \quad k=2,3,\ldots,N-1. \end{aligned} \tag{23}$$

It is worth noting that the metric $g^{(D)}$ is independent of θ_{N-1}.

Now in order to calculate $g^{(C)}$, we define $\chi_{\alpha_m}^{(m)}$ ($\alpha_m = 1,2,\ldots,2(m-1)$) as the $2(m-1)$ real parameters of the coset component $\Omega^{(m;N)}$ ($m=2,\ldots,N$) such that

$$\chi_{\alpha_m}^{(m)} = \begin{cases} \gamma_{\alpha_m}^{(m)} & \text{for} \quad \alpha_m = 1,\ldots,m-1, \\ \xi_{\alpha_m}^{(m)} & \text{for} \quad \alpha_m = m,\ldots,2(m-1). \end{cases} \tag{24}$$

Then the infinitesimal Bures distance over the coset coordinates can be written as

$$\begin{aligned} \sum_{i<j}^{N} \Lambda_{ij} |(\Omega^\dagger \, \mathrm{d}\Omega)_{ij}|^2 &= \sum_{m=2}^{N} \sum_{m'=2}^{N} \sum_{\alpha_m=1}^{2(m-1)} \sum_{\beta_{m'}=1}^{2(m'-1)} g_{\alpha_m,\beta_{m'}}^{(m,m';N)} \, \mathrm{d}\chi_{\alpha_m}^{(m)} \, \mathrm{d}\chi_{\beta_{m'}}^{(m')} \\ &= \sum_{m=2}^{N} \sum_{\alpha_m,\beta_m=1}^{2(m-1)} g_{\alpha_m,\beta_m}^{(m,m;N)} \, \mathrm{d}\chi_{\alpha_m}^{(m)} \, \mathrm{d}\chi_{\beta_m}^{(m)} \\ &\quad + 2 \sum_{m<m'}^{N} \sum_{\alpha_m=1}^{2(m-1)} \sum_{\beta_{m'}=1}^{2(m'-1)} g_{\alpha_m,\beta_{m'}}^{(m,m';N)} \, \mathrm{d}\chi_{\alpha_m}^{(m)} \, \mathrm{d}\chi_{\beta_{m'}}^{(m')}. \end{aligned} \tag{25}$$

Now invoking the decomposition

$$\Omega = \Omega^{(N;N)} \Omega^{(N-1;N)} \cdots \Omega^{(2;N)}, \tag{26}$$

we can write

$$|(\Omega^\dagger \, \mathrm{d}\Omega)_{ij}|^2 = \sum_{m=2}^{N} |(K^{(m;N)})_{ij}|^2 + 2 \sum_{m<m'}^{N} \mathrm{Re}\{(K^{(m;N)})_{ij}(K^{(m';N)})_{ij}^*\}, \tag{27}$$

where we have defined

$$K^{(m;N)} = W^{(m;N)\dagger}(\Omega^{(m;N)\dagger} \, \mathrm{d}\Omega^{(m;N)}) W^{(m;N)}, \tag{28}$$

with

$$W^{(m;N)} = \Omega^{(m-1;N)} \cdots \Omega^{(3;N)} \Omega^{(2;N)} \tag{29}$$

and $W^{(2;N)} = 1$. The first term of the rhs of equation (27) shows sum over all pure contribution of each coset component in the Bures distance, and can be identified with

$$\sum_{i<j}^{n} \Lambda_{ij} |(K^{(m;N)})_{ij}|^2 = \sum_{\alpha_m,\beta_m=1}^{2(m-1)} g_{\alpha_m,\beta_m}^{(m,m;N)} \, \mathrm{d}\chi_{\alpha_m}^{(m)} \, \mathrm{d}\chi_{\beta_m}^{(m)}. \tag{30}$$

The second term, however, shows the sum over mixed contribution of all pairs of coset components in the Bures distance and can be used to write

$$\sum_{i<j}^{n} \Lambda_{ij} \operatorname{Re}\{(K^{(m;N)})_{ij}(K^{(m';N)})_{ij}^*\} = \sum_{\alpha_m,=1}^{2(m-1)} \sum_{\beta_{m'}=1}^{2(m'-1)} g_{\alpha_m,\beta_{m'}}^{(m,m';N)} \, d\chi_{\alpha_m}^{(m)} \, d\chi_{\beta_{m'}}^{(m')}. \tag{31}$$

Therefore $g^{(C)}$ is defined by the following symmetric matrix

$$g^{(C)} = \begin{pmatrix} g^{(2,2;N)} & g^{(2,3;N)} & \cdots & g^{(2,N;N)} \\ & g^{(3,3;N)} & \cdots & g^{(3,N;N)} \\ & & \ddots & \vdots \\ & & & g^{(N,N;N)} \end{pmatrix}. \tag{32}$$

Now the object is to calculate the matrix elements of $g^{(C)}$, which can be achieved if we can find an expression for $(K^{(m;N)})_{ij}$ of equations (30), (31) in terms of the coset coordinates $\chi_{\alpha_m}^{(m)}$ (or equivalently in terms of $\gamma_{\alpha_m}^{(m)}$ and $\xi_{\alpha_m}^{(m)}$). Therefore we have to expand $\Omega^{(m;N)\dagger} \, d\Omega^{(m;N)}$ of equation (28) in terms of $d\gamma^{(m)}$ and $d\xi^{(m)}$. This can be achieved by using the factorization $\Omega^{(m;N)} = X^{(m;N)} R^{(m;N)} X^{(m;N)\dagger}$ where we get

$$\begin{aligned} \Omega^{(m;N)\dagger} \, d\Omega^{(m;N)} &= X^{(m;N)} R^{(m;N)T} X^{(m;N)\dagger} (dX^{(m;N)}) R^{(m;N)} X^{(m;N)\dagger} \\ &\quad - (dX^{(m;N)}) X^{(m;N)\dagger} + X^{(m;N)} R^{(m;N)T} (dR^{(m;N)}) X^{(m;N)\dagger}. \end{aligned} \tag{33}$$

By knowing that $X^{(m;N)}$ is a unitary diagonal matrix and $R^{(m;N)}$ is an orthogonal matrix, the first two terms can be calculated easily by using the relation

$$dX_{ij}^{(m;N)} = i e^{i\xi_i^{(m)}} \delta_{ij} \, d\xi_i^{(m)}. \tag{34}$$

On the other hand to calculate the third term, equation (15) may be used to show that

$$dR_{ij}^{(m;N)} = \sum_{k=1}^{m-1} \Gamma_{ij;k}^{(m,N)} \, d\gamma_k^{(m)}, \tag{35}$$

where for a fixed $k (= 1, 2, \ldots, m-1)$, the $\Gamma_{ij;k}^{(m,N)}$ denote the matrix elements of an $N \times N$ matrix with nonzero elements:

$$\Gamma_{ij;k}^{(m;N)} = \left(\hat{\gamma}_i^{(m)} \delta_{jk} + \hat{\gamma}_j^{(m)} \delta_{ik} - 2\hat{\gamma}_i^{(m)} \hat{\gamma}_j^{(m)} \hat{\gamma}_k^{(m)}\right)(\cos\gamma^{(m)} - 1)/\gamma^{(m)} - \hat{\gamma}_i^{(m)} \hat{\gamma}_j^{(m)} \hat{\gamma}_k^{(m)} \sin\gamma^{(m)}$$
$$\text{for} \quad 1 \leqslant i, j \leqslant m-1$$
$$\Gamma_{im;k}^{(m;N)} = -\Gamma_{mi;k}^{(m;N)} = \left(\delta_{ik} - \hat{\gamma}_i^{(m)} \hat{\gamma}_k^{(m)}\right) \sin\gamma^{(m)}/\gamma^{(m)} + \hat{\gamma}_i^{(m)} \hat{\gamma}_k^{(m)} \cos\gamma^{(m)} \tag{36}$$
$$\text{for} \quad 1 \leqslant i \leqslant m-1$$
$$\Gamma_{mm;k}^{(m;N)} = -\hat{\gamma}_k^{(m)} \sin\gamma^{(m)}.$$

Using equations (34) and (35) in equation (33), we get

$$(\Omega^{(m;N)\dagger} \, d\Omega^{(m;N)})_{ij} = \exp\{i(\xi_i^{(m,N)} - \xi_j^{(m,N)})\} \sum_{k=1}^{m-1} \left((E_{ij}^{(m;N)})_{\gamma_k} \, d\gamma_k^{(m)} + i(E_{ij}^{(m;N)})_{\xi_k} \, d\xi_k^{(m)}\right), \tag{37}$$

where $(E_{ij}^{(m;N)})_{\gamma_k}$ and $(E_{ij}^{(m;N)})_{\xi_k}$ have been defined by

$$\left(E_{ij}^{(m;N)}\right)_{\gamma_k} = \sum_{l=1}^{m} R_{li}^{(m;N)} \Gamma_{lj;k}^{(m;N)} \tag{38}$$

$$\left(E_{ij}^{(m;N)}\right)_{\xi_k} = R_{ki}^{(m;N)} R_{kj}^{(m;N)} + \delta_{ij}\delta_{jk}. \tag{39}$$

Therefore for $K^{(m;N)}$ of equation (28), we find the following matrix elements

$$K_{ij}^{(m;N)} = \sum_{r=1}^{m-1} \left\{ \left(K_{ij}^{(m;N)}\right)_{\gamma_r} d\gamma_r^{(m)} + \left(K_{ij}^{(m;N)}\right)_{\xi_r} d\xi_r^{(m)} \right\}, \tag{40}$$

where we have defined

$$\left(K_{ij}^{(m;N)}\right)_{\gamma_r} = \sum_{k,l=1}^{m} \left(W_{ik}^{(m;N)}\right)^* \left(W_{lj}^{(m;N)}\right)\left(E_{kl}^{(m;N)}\right)_{\gamma_r}. \tag{41}$$

and

$$\left(K_{ij}^{(m;N)}\right)_{\xi_r} = \sum_{k,l=1}^{m} \left(W_{ik}^{(m;N)}\right)^* \left(W_{lj}^{(m;N)}\right)\left(E_{kl}^{(m;N)}\right)_{\xi_r}, \tag{42}$$

Finally, putting everything together, we find the following formula for the matrix elements of the metric $g^{(C)}$:

$$g_{\gamma_r^{(m)}\gamma_s^{(m')}}^{(m,m';N)} = \sum_{i<j}^{N} \Lambda_{ij} \operatorname{Re}\left\{ \left(K_{ij}^{(m;N)}\right)_{\gamma_r^{(m)}} \left(K_{ij}^{(m';N)}\right)_{\gamma_s^{(m')}}^* \right\}, \tag{43}$$

$$g_{\xi_r^{(m)}\xi_s^{(m')}}^{(m,m';N)} = \sum_{i<j}^{N} \Lambda_{ij} \operatorname{Re}\left\{ \left(K_{ij}^{(m;N)}\right)_{\xi_r^{(m)}} \left(K_{ij}^{(m';N)}\right)_{\xi_s^{(m')}}^* \right\}, \tag{44}$$

$$g_{\gamma_r^{(m)}\xi_s^{(m')}}^{(m,m';N)} = \sum_{i<j}^{N} \Lambda_{ij} \operatorname{Re}\left\{ \left(K_{ij}^{(m;N)}\right)_{\gamma_r^{(m)}} \left(K_{ij}^{(m';N)}\right)_{\xi_s^{(m')}}^* \right\}, \tag{45}$$

$$g_{\xi_r^{(m)}\gamma_s^{(m')}}^{(m,m';N)} = \sum_{i<j}^{N} \Lambda_{ij} \operatorname{Re}\left\{ \left(K_{ij}^{(m;N)}\right)_{\xi_r^{(m)}} \left(K_{ij}^{(m';N)}\right)_{\gamma_s^{(m')}}^* \right\}, \tag{46}$$

for $r = 1, 2, \ldots, m-1$, $s = 1, 2, \ldots, m'-1$ and $m, m' = 1, 2, \ldots, N$. Since the eigenvalues coordinates are just included in the Λ_{ij} terms of equations (43)–(46), it is clear that all matrix elements of the metric $g^{(C)}$ are simply sum of the products of two independent functions, the eigenvalues coordinates and the coset coordinates. It should be noted that for real density matrices, i.e. $\xi_i^{(m)} = 0$ for $i = 1, 2, \ldots, m-1$ and $m = 1, 2, \ldots, N$, we have $\Omega^{(m;N)} = R^{(m;N)}$. In this case all quantities are real and equation (43) gives all matrix elements of the $N(N-1)/2$-dimensional metric $g^{(C)}$.

4. Conclusion

We provide a method for explicit computation of the Bures metric in N-level quantum systems, based on the coset parametrization of density matrices. We show that in the canonical coset parametrization, the Bures metric matrix is divided into two blocks, an $(N-1)$-dimensional diagonal matrix corresponding to the eigenvalues coordinates, and an $N(N-1)$-dimensional matrix corresponding to the coset coordinates. It also provides a factorization of the Bures measure on the space of density matrices as the product of the measure on the space of eigenvalues and the truncated Haar measure on the space of unitary matrices.

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